Nested instantons have been recently introduced in physics as objects describing the parabolic reduction of gauge connections at punctures on Riemann surfaces and are understood in terms of moduli spaces of stable representations of certain quivers. The relevant physical quantities compute some of their virtual invariants, which in some case have a connection to the cohomology of character varieties. We will study the moduli spaces of stable representations of a class of nested instantons quivers and show how these are isomorphic to moduli spaces of (suitably defined) flags of framed torsion-free sheaves on the projective plane, reducing in rank one to nested Hilbert schemes of points on the complex plane. The existence of a perfect obstruction theory and the quiver description enable us to compute virtual invariants of flags of sheaves and to generalize their computation to nested Hilbert schemes of points on (smooth) toric surfaces.