Sparse expanders have negative curvature

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We prove that bounded-degree expanders with non-negative Ollivier-Ricci curvature do not exist, thereby solving a long-standing open problem suggested by Naor and Milman and publicized by Ollivier (2010). In fact, this remains true even if we allow for a vanishing proportion of large degrees, large eigenvalues, and negatively-curved edges. To establish this, we work directly at the level of Benjamini-Schramm limits, and exploit the entropic characterization of the Liouville property on stationary random graphs to show that non-negative curvature and spectral expansion are incompatible "at infinity". We then transfer this result to finite graphs via local weak convergence. The same approach applies to the Bakry-Émery curvature condition $\text{CD}(0, \infty)$, thereby settling a recent conjecture of Cushing, Liu and Peyerimhoff (2019).